Length and area: Key characters in the curriculum story

Tom Francome explores perimeter and area.

Figure 1: Which track is longer? You might like to consider how you would justify this before reading on.

(All images from LUMEN curriculum: 7.6 Length and Area www.lboro.ac.uk/services/lumen/curriculum/)

Mathematics teachers often have questions about perimeter and area: should you teach them together or separately initially? What tasks should be included and how should they be sequenced? What order should different shapes appear? When should we worry about units? I had considered these issues as a teacher, but in developing the LUMEN curriculum, I had an opportunity to look with fresh eyes at how the story might unfold. I found it helpful to think of the central characters as being length and area rather than perimeter and area. In this article, I explore how those central characters might develop.

You can think of curriculum as a story that unfolds with elements like characters, action, actors, setting and plot, having mathematical equivalents (Dietiker, 2015). Mathematical characters (such as numbers, expressions, or shapes) are the objects of the story, mathematical action is manipulating a character (for example, by adding one to the base of a rectangle and subtracting one from its height), mathematical actors are who performs the action (for example, student, teacher, resource author) mathematical setting is the representation in which the action takes place, such as physical shapes on a table, dynamic geometry software, or shapes drawn on a square grid.

It is tempting to think of perimeter and area as *just* properties of 2-D shapes and start thinking about which shapes you should begin calculating with. For example, English curriculum materials often start with a square but maybe it is better to consider the more general rectangle before considering the square as a special case: "rules first, exceptions later" (Hewitt, 1994, p. 41)? Is it better to think of a triangle as half a rectangle or half a parallelogram? These questions are important but skipping to them potentially misses out on important development of the underpinning concepts of length and area.

Focus on length before perimeter

Length is an important character in the curriculum story. A significant challenge of the secondary curriculum is to help learners engage with such important concepts in a meaningful way. With length (and area), the key action is comparison either to a given unit (measurement) or another length. For example, the task in Figure 1, is to make a comparison between the lengths of track. In this, and other, situations, looks can be deceiving and that is where we need mathematics to shift from 'might be' to 'must be' (see Figure 9). Figure 1 is an optical illusion; the tracks are congruent but appear different as the natural tendency is to compare the closest lengths (the bottom edge of the top track and the top edge of the bottom in Figure 1). The width of the track means these edge lengths differ even though the tracks are congruent; it can be fun with learners to swap the tracks over – then the largest seems to switch. Learners can be asked to reason about what is happening. The issue can be resolved by measuring, placing the tracks on top of each other as in Figure 9, or flipping the lower one over so the congruent edges align.

There are many misconceptions around from early experiences of length measurement (see Clements, 1999 for more detail). For example, learners can think that if endpoints align then the lengths are equal (an issue in Figure 2a extended in 2f, 2g, 2h, 2i, 2j) or that a length split into more pieces is longer, or that a length split into fewer pieces is longer since each piece is longer. I'd encourage you to engage with Figure 2 before moving on and try to observe what you are taking for granted.

Try this...

For each pair, is there more red ink or blue ink?
You could trace them to decide.

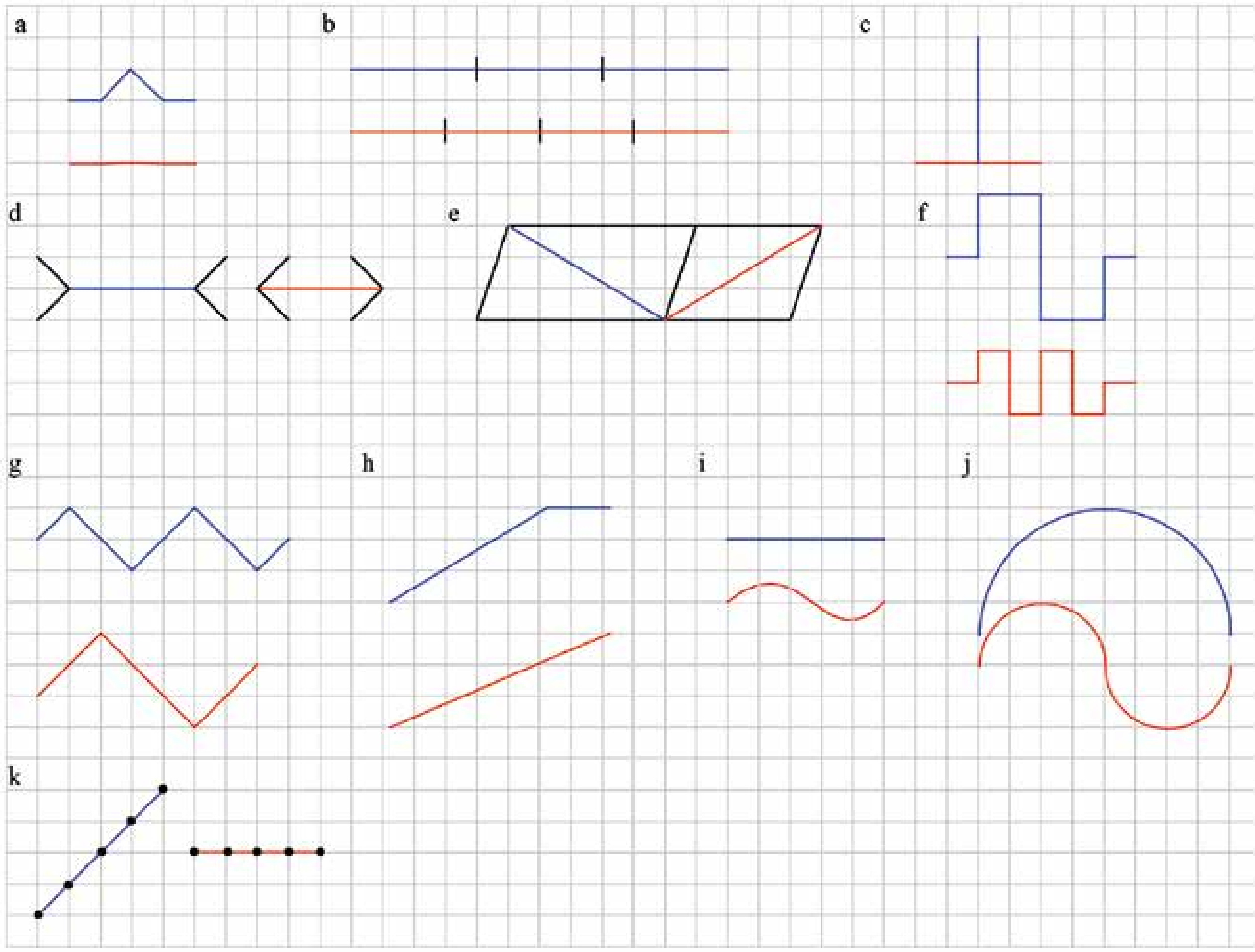

Figure 2: An activity designed to work on the concept of length and encourage both reasoning and embodiment via the tracing. You may find some surprises here if you have not considered similar questions before.

Learners who find this straightforward can be challenged to consider the misconception each question is there to surface – that is, if someone chose the incorrect length, what would they have to be thinking to do what they are doing? Learners may consider this more obvious in Figure 2b but less obvious in 2k. Throughout this exercise, learners are encouraged to trace the lengths with their finger, to support the reasoning, develop the sense of what length is, and to exploit the tracing effect – that running a finger over elements of geometry examples often enhances learning (See, for example, Hu *et al.*, 2015).

Some teachers I have used this task with have suggested that learners would not have sufficient background knowledge to reason with 2j unless they knew circle formulae. However, 2j is structurally similar to both 2f and 2g. In 2f, learners can reason using the unit gridlines that there are twice as many of the step-shapes, but each one is half as long. In 2g, reasoning can rely on a different unit, the diagonal of the grid square , as both the red and blue lines are made entirely of this unit, they can be compared directly by counting. The tasks further play into the misconception of aligning endpoints. With 2j, the unit can be seen as the semicircle: the blue semicircle is the red length scaled by a factor of 2, but in the red, there are twice as many semi-circular units. 2k is a variation of 2b but without the aligning endpoints. This foreshadows later work with different units. Both the lines have length 4, but 4 what? Introducing non-standard units too early can interfere with the development of measurement concepts. Nunes et al. (1999) found that measuring with rulers benefited learners, as they offer a representation of number analogous to the number line. However, introducing non-standard units at Key Stage 3 offers an opportunity for reasoning that can be grounded in learners' understanding of more common units like centimetres. Learners can appreciate the need for standard units because they can build on their previous concepts of measurement.

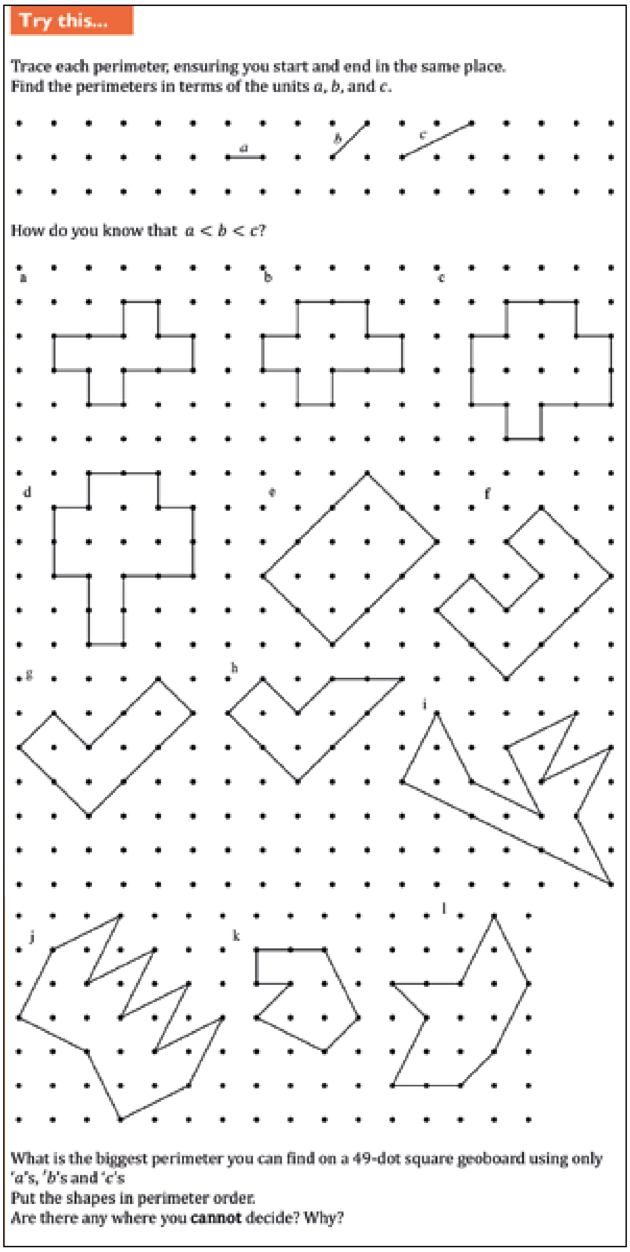

Figure 3: A task to practise perimeter/length concepts and the subordinate practice of collecting terms.

Perimeter

Conceptually, perimeter can build on the notions of length and measurement. The word perimeter stems from Greek *perimetros*, based on *peri*- 'around' and *metron* 'measure'. However, work here can still focus on units and serve as a context to revisit collecting like terms (See Foster, C. 2021) by combining different units within the same task. Drawing shapes on a square geoboard using combinations of the units a, b and c, as in Figure 3, offers further opportunities for reasoning, alongside practice of the perimeter/length concepts and the subordinate practice of collecting terms. For example, how do you know that $a + b > c$, or $2b > c$?

So far, we have focused on a *square* grid, which I had perhaps previously taken for granted. However, considerations of perimeter can be made on an *isometric* grid, as in Figure 4. This focus on units can be further extended to thinking about area.

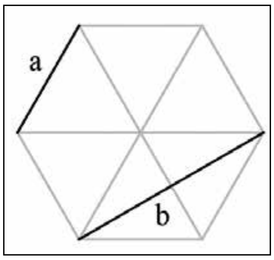

Figure 4: On a grid made of six equilateral triangles, a and b are two different lengths.

Draw some polygons and order them in terms of their perimeters.

What is Area?

Teacher: What is area?

Student: Base times height?

Readers might like to reflect on this interaction. How would you answer the pupil? What are the strengths and weaknesses of the pupil's answer?

How do you think about what area is? Conceptually, length and volume/capacity seem easier to tackle than area, which is somewhat more nebulous. You might be thinking about the number of square units a shape has. 'Space' is sometimes reserved for 3D so you might think of a measure of the amount of 'surface' a shape covers. But how do you conceptualise this? It might be thought of as how much paint it would need to cover it or as how long it takes to shade in. Shading can give an intuitive sense of area that can be contrasted with tracing the perimeter. Another way to think of area is as the number of tiles that could cover the shape (e.g., how many square stickers to cover a Rubik's cube). We tend to do this with square grids, but some work with different grids can help learners accept why this is a sensible, but arbitrary, choice.

Figure 5 and Figure 6 contain two similar tasks about comparing area. The first keeps the *side length* the same between grids. The second keeps the *tile area* (square, triangle, rhombus, hexagon) the same between grids. In Figure 5, we are not expecting learners to know how to calculate the relative areas. However, we do hope to surface the issue that it is more straightforward to compare areas that have the same units. (These tasks were inspired by Hewitt's *Greatest and Smallest* from the RLDU Area and Perimeter pack.) The idea is to compare shapes within pairs initially. This relies on the familiar reasoning that area is conserved if you cut a shape up and move the pieces around without them overlapping. However, comparing between different grids is far less straightforward, for example shapes a_1, c_1, and d_1 all cover 9 grid tiles, but does it make sense to say they all have an area of 9? This issue is later addressed by considering the triangle with side length 1 as having an area of 9. Defining the area of one triangular tile as T might feel less strange than calling the area 1 (if the triangle side length is 1 unit, then $T = \frac{\sqrt{3}}{2}$ sq. units, and the other shapes can be similarly approached).

The shapes and grids are chosen so that learners can reason about their areas without needing these calculations. For example, once we know that the tile side lengths are equal between grids, we can conjecture that the area of $d_1 < c_1$. A discussion point is what happens to the area if you fix the four side lengths of a rhombus and vary the angles (this can be contrasted with shearing, see Shore *et al.*, 2023). Learners can reason that the rhombus is smaller – the square is the maximum area and the rhombus with the same sides can take any area between this maximum and zero, when the rhombus collapses to a line segment. Learners can reason that the hexagon tile is made up of six triangles, and the rhombus is made of two triangles. This allows comparisons to be made between shapes on different grids. However, whilst it is possible to reason that $d_1 < c_1$ it is harder to justify that $d_1 < c_2$ without calculation. This kind of task can help learners appreciate both the concept of area as a measure of enclosed surface, and the utility of a common unit.

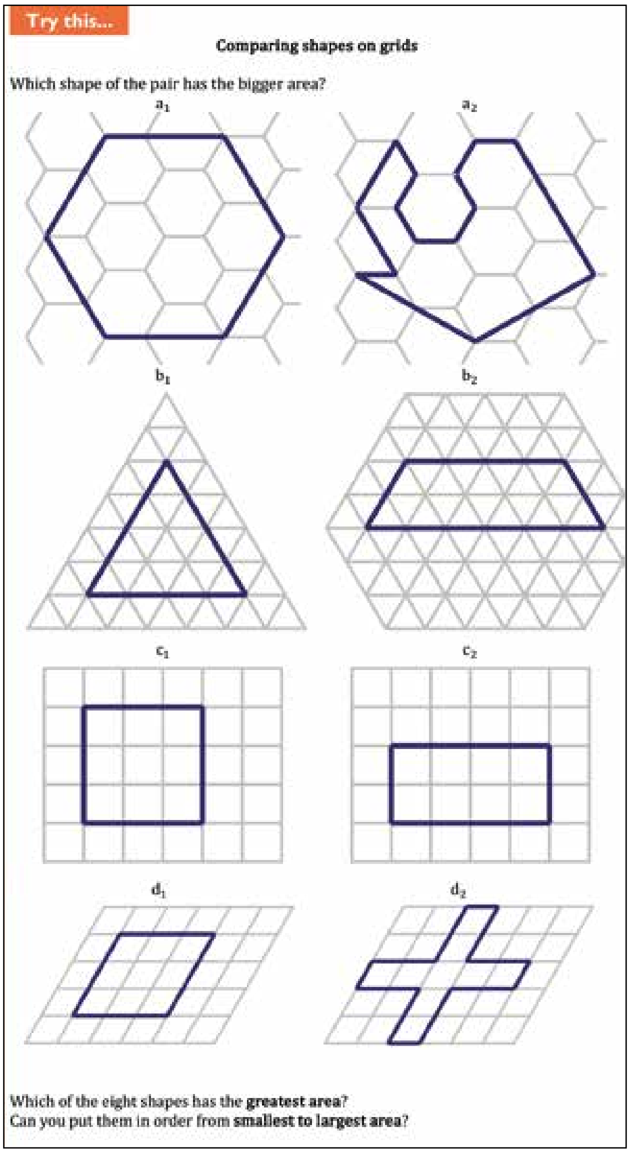

Figure 5: Comparing the areas of shapes on different grids. In this case, the unit length is equal across grids.

You might reasonably ask 'what if not… equal side lengths but equal areas?' Scaling the grids so this is true, as in Figure 6, produces quite different diagrams and vastly simplifies comparison, reducing this to counting. Comparing the area of 2-D shapes is much more straightforward when they are on the same grid and compared using the same unit. Square units are the most common way to measure area, because they tesselate and their areas are easy to calculate. However, these kinds of task raise awareness of the area unit and that this is a choice rather than the only option.

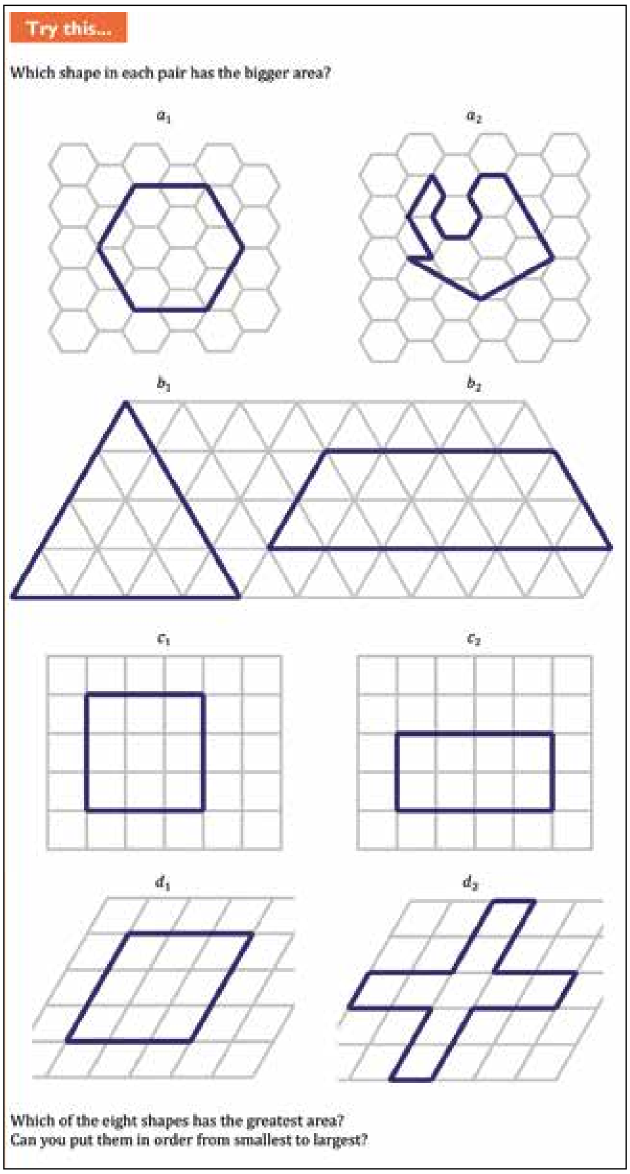

Figure 6: Comparing the areas of shapes on different grids. In this case, the area of each tile is equal across grids.

It is only after using the square grid to estimate areas that the more familiar territory of areas of rectangles and rectilinear shapes, parallelograms, triangles, and trapeziums is considered. The aim throughout is that this work gradually unfolds with sufficient detail and sufficiently stimulating tasks to interest those who have met ideas before as well as to enable those with gaps to really understand the important ideas of the story. As we move away from the grid structure, there is an emphasis on discussing and understanding what would need to be measured to calculate the area of a given shape, as in Figure 7.

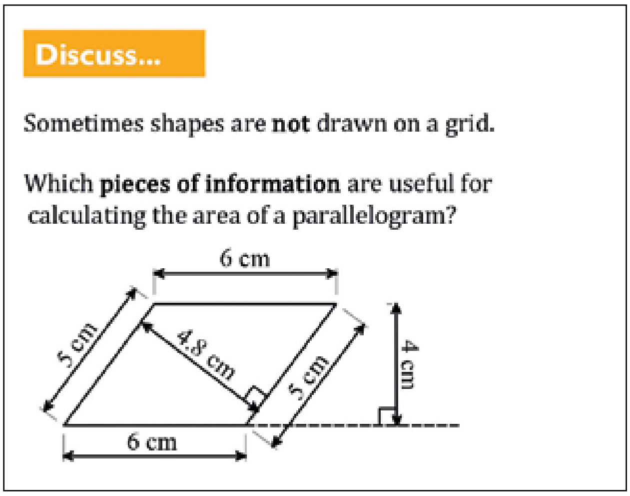

Figure 7: Part of a discussion task on what matters in finding the area of a parallelogram.

Try this...

Choose 5 of the **trapeziums** given above.

For each one, draw:

a. A **triangle** with the same area as the trapezium.
b. A **parallelogram** with the same area as the trapezium.
c. A **rectangle** with the same area as the trapezium.
d. A **triangle** with **half** of the area of the trapezium.
e. A **parallelogram** with **twice** the area of the trapezium.
f. A **rectangle** with **half** of the area of the trapezium.
g. **Two** triangles where **the difference** between the areas is **the same** as the area of the trapezium.

Which of these (a-g) do you find easiest? Why?
Can you describe a rule for how to do it each time?

Figure 8: An exercise that allows learners to compare directly the relationships between the areas of common shapes.

Parallelograms can always be sheared into rectangles and triangles are conceptualised as half of a parallelogram. This journey allows readers to repeatedly recap previous ideas (such as the area of the parallelogram) through later ideas. Interleaving the ideas within the same tasks, such as Figure 8, also supports learners to discriminate between them. The character of triangle area is further developed by considering what the half might mean in a formula such as ‘half base times height’. This supports later work considering different conceptions of the area of a trapezium. Trapezia (or ‘trapeziums’) are initially a context to practise the area of a triangle but, before this is offered, there is an opportunity for learners to think for themselves about different ways to find a trapezium’s area (Francome, 2016).

Figure 9: Which track is actually longer? Looks can be deceiving, which is why we need mathematical ideas like measurement to compare lengths.

Area and perimeter can often be reduced to 'finding the area/perimeter of particular shapes', so learners work on '*calculating* the area of a triangle' rather than 'the area of a triangle' and skip over important character development of both the shapes and the concepts of area and perimeter themselves. Thinking about length and area as the protagonists in the story has helped me to ponder the mathematics more deeply and I hope it will do the same for both you and your learners.

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