Being flexible about division

Colin Foster explores options to support division strategies

I recently saw a child trying to work out 45 divided by 5. They wrote:

$$\begin{array}{r} 0\phantom{^{4}5} \\ 5\overline{)\not{4}\;{}^{4}5} \end{array}$$

saying, "Five into four is zero, remainder four. Five into forty-five is …" – and they didn't know. Indeed, if they had known this, then perhaps they wouldn't have been using this short-division method. The method was just returning them to the same question that they started with.

The method is supposed to help you with large numbers by breaking up the number into separate digits, so why doesn't it work here? This problem always arises whenever a (single-digit) divisor is greater than the leading digit of the (two-digit) dividend (Figure 1). Applying the method merely reproduces the original division. It seems as though you have to go about it some other way, such as skip-counting up in 5s from 0 to 45, or doubling 45 and dividing by 10.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} \qquad\qquad \begin{array}{r} \text{quotient} \\ \text{divisor}\overline{)\text{dividend}} \end{array}$$

Figure 1. A reminder of the terms dividend, divisor and quotient.

While I was thinking about how I might try to help, the child said something like, "Oh I see what I've done wrong. It's not four – it's for<u>ty</u>!" They then pondered "Five into forty", which they deduced was twice "ten into forty", and so must be 8. Then, "Five into five is one" and they wrote:

$$\begin{array}{l} 8+1=9 \\ 5\overline{)4\;5} \end{array}$$

and declared that the answer is 9. They seemed to think that this is what they should have been doing all along, and that saying 'five into four', rather than 'five into forty' was simply a mistake. To me this showed a lot of understanding about what is going on with division and a nice flexibility in using the algorithm. Perhaps it might have been a bit nicer to write both the 8 and the 1 in the 1s column, as

$$\begin{array}{r} 1 \\ 8 \\ 5\overline{)4\;\;5} \end{array}$$

This might be more natural when writing the division algorithm 'upside down' (from a UK perspective), as:

$$\begin{array}{r} 5\underline{)4\;\;5} \\ 8 \\ \underline{+\;1} \\ \underline{\;\;\;9} \end{array}$$

(Of course, this layout is perfectly usual in many countries.)

Generalising the approach

The child's approach can be used more generally. So, I asked the child if they could do $\frac{48}{6}$. Using the algorithm in the usual way ('six into four'), we are left with the same division that we began with, $\frac{48}{6}$, just as happened with $\frac{45}{5}$. But if we instead consider 'six into forty', that doesn't seem to help this time, because 6 isn't a factor of 40. However, it would be possible to partition the 48 differently.

It was convenient before to split as $45 = 40 + 5$, because 40 was an 'easy' multiple of 5. But, with division by 6, the partition $48 = 40 + 8$ is less convenient, since 40 isn't a multiple of 6.

Do we happen to know any multiples of 6 nearish to 48? Note that there is no absolute requirement to find the largest multiple of 6 less than 48. It seems unlikely that the child would happen to know this, if they didn't know which multiple 48 was of 6 (Note 1). Perhaps by analogy with $\frac{45}{5}$, the child eventually went for a multiple of 10, this time 30, because they could figure out that $\frac{30}{6}$ was equal to 5. They wrote:

$$\begin{array}{l} 5+3=8 \\ 6\overline{)^{3}\not{4}\;\;{}^{1}8} \end{array}$$

But it could equivalently be expressed as:

$$\begin{array}{r} 6\overline{)4\ 8} \\ 5 \\ +\ 3 \\ \hline 8 \\ \hline \end{array}$$

To me, although probably not for the child, what's going on here is clearer when written as fractions:

$$\frac{48}{6} = \frac{30 + 18}{6} = \frac{30}{6} + \frac{18}{6} = 5 + 3 = 8$$

This method relies on repeatedly pulling out known (or easily found) multiples of the divisor by inspection, so reducing the original division to an easier one at each stage. Aren't short or long division just slightly more formalised versions of this essential process?

Division algorithms

My relationship with long division over the years has been complicated. There was a time when I would have said that long division isn't the kind of mathematics that I think is important. Indeed, there was a time when I might have said, "I don't teach long division", and focused on other methods of division instead. But I've changed my view, and now I think that there is a lot of interesting mathematics in long division – so much so that I actually like it and I think I would teach it even if it weren't on the curriculum!

It feels to me that methods like long division emerge quite naturally in this kind of context, and you can hold on to the relational thinking (Skemp, 1976) much further than I had previously thought realistic. Suppose that a child is comfortable with saying $\frac{6}{2} = 3$ and $\frac{6}{3} = 2$, relating these facts to $2 \times 3 = 6$ and $3 \times 2 = 6$. And they have reasonable facility with some of the easier multiplication tables. I think it isn't far from there to short and long division while 'keeping it relational' and not descending into arbitrary steps that must be done in a certain way purely because the teacher says so. Of course, if you are prepared to go 'instrumental' instead, then you can just tell them the process and make them do it step by step. But here I'm interested in staying with a good understanding of what it all means at each point as we go.

Big multiples

The first thing I would try to do is get from $\frac{6}{2} = 3$ to $\frac{60}{2} = 30$ and $\frac{600}{2} = 300$. It can be easier to start with 600 than 60, because the natural language of 'six hundred' makes it sound more like you have six of something (which you do), and so when you divide by two you must have three of those things. For 60 it might be helpful to say "six tens". For division by 2, children may say "half of", but I would try to stick with the language of "divided by 2", because this generalises to other divisors that we will want to go on to use.

Can they work out $\frac{666}{2}$? If they are happy that $\frac{600}{2} = 300$ and $\frac{60}{2} = 30$ and $\frac{6}{2} = 3$, then I find that the jump to $\frac{666}{2} = 333$ is not too difficult. When we say that multiplication is distributive over addition, but division isn't, we mean that although $a(b + c) = ab + ac$, with division $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$. However, I find that children will naturally correctly assume that $\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a}$. We seem to just know from experience that if you have several things to share out evenly, then if you share out some of them evenly, and then share out the rest evenly, then you've shared them all out evenly.

It can be fun to deal with very big numbers and see that big numbers (even ones that the child might not yet be able to say in words) don't necessarily make things more difficult:

$$\frac{666{,}666{,}666}{2} = 333{,}333{,}333$$

Then I would try $\frac{606}{2}$ and $\frac{660}{2}$.

Then it's time to change the 6s into 8s. So, the child can do things like $\frac{800}{2}$ and $\frac{808}{2}$. Then we could try $\frac{806}{2}$ and slip in other ones, like $\frac{826}{2}$. I would try to make these look as complicated as possible, while sticking with division by 2 and using only even digits in the dividend. Get the child doing things like $\frac{6{,}840{,}602}{2}$. I have found that children think that this is easy and rather enjoy the very large numbers. Throughout all of this, I want to avoid telling them that 'You just divide each digit by 2'. If they observe that for themselves, then that's fine. But I don't want them to be following a rule that I've given them. I want them to stay with a sense of the place value, so that when they divide the 4 by 2 they know that they are dividing 40,000 by 2. And I'd ask them about this kind of thing from time to time as they go. ("What does that 4 there mean?")

Note that with divisions like this there is of course no obligation to work right-to-left. Left-to-right is just as good (see Simons, 2024) – or we could begin in the middle and go in any order we wish. Alternatively, perhaps we'd rather do all the 8s, then all the 6s, then the 4s, then the 2s, then the zeros.

Then I think I would change the divisor to 3. They know $\frac{6}{3} = 2$, so I would ask them what divisions they can do using that fact. It's their job to see what doors this opens. They will come up with things like $\frac{600}{3}$ and, eventually, divisions like $\frac{60\,933}{3}$. However, they may well end up inventing some that contain digits in the dividend that are not multiples of 3. I might just note for now that these are harder, and we'll come back to them later. I would park them on another piece of paper, to be revisited later. I don't want them to think that anything is ever 'beyond them' – just that we might do some things later, instead of immediately.

I think it's good to note the power of what the child can do by this point, dividing arbitrarily large (but carefully chosen) numbers by 2 or 3. This can be extended to division by 4, 5, 6 and higher numbers, but the possibilities become increasingly limited to do so without any remainders arising. Let the child explore what possibilities they can come up with. With a divisor of 6, for example, they are restricted to 6s and 0s digits in the dividend, and all of the answers will be strings of 0s and 1s. It feels to me useful to spend plenty of time on this, before getting into remainders. This creates a 'need' for handling remainders, as the list of divisions that we're saving for later begins to grow. The child I was working with was desperate to know what to do if some of the digits 'didn't go'!

Remainders

How can division get any harder than what we have done so far? There is only one way in which this can happen, and that is to have remainders. Remainders are the only things that can make any division harder than these. Once you figure out how remainders work, then you can do any division problem.

So, for me, the big thing to focus on is: What is a remainder actually?

Let's take a more difficult division; one that we couldn't do before: $\frac{471}{3}$. Conceptually, the way I am thinking about this is the way I've written it below, but I wouldn't write it out like this for the child. I would initially stay with whatever way of writing it they seem most comfortable with. But this is how I am thinking about what's happening:

$$\frac{471}{3} = \frac{400}{3} + \frac{70}{3} + \frac{1}{3}$$

$$= \frac{300}{3} + \frac{100}{3} + \frac{70}{3} + \frac{1}{3}$$ Decompose the 400 into 300 + 100.

$$= 100 + \frac{170}{3} + \frac{1}{3}$$ Combine the remainder 100 with the 70.

$$= 100 + \frac{150}{3} + \frac{20}{3} + \frac{1}{3}$$ Decompose the 170 into 150 + 20.

$$= 100 + 50 + \frac{21}{3}$$ Combine the remainder 20 with the 1.

$$= 100 + 50 + 7$$

$$= 157$$

If the child suggests taking out multiples of 3 that are smaller than these ones, that's fine – it may just take a little longer, but the answer will come out correctly anyway. And you can always go back later and look for bigger multiples to remove, which saves steps. So I would initially go with whatever they suggest, and avoid implying that there are some rules or conventions that I haven't shared around what they are supposed to be doing. If there is a remainder left at the very end, then we simply arrive at a non-integer answer, but there is no mystery about the remainder and what it is. For example,

$$\frac{47}{3} = \frac{40}{3} + \frac{7}{3}$$
$$= \frac{30}{3} + \frac{10}{3} + \frac{7}{3}$$
$$= 10 + \frac{17}{3}$$
$$= 10 + \frac{15}{3} + \frac{2}{3}$$
$$= 10 + 5 + \frac{2}{3}$$
$$= 15\frac{2}{3}$$

Alternatively, if you can find a closer multiple of 3, then

$$\frac{47}{3} = \frac{45}{3} + \frac{2}{3}$$
$$= 15\frac{2}{3}$$

Or you could even go too far and have to come back:

$$\frac{47}{3} = \frac{48}{3} + \frac{1}{3}$$
$$= 16 - \frac{1}{3}$$
$$= 15\frac{2}{3}$$

It is not absolutely necessary to find the greatest multiple less than the dividend. I think of this as a polishing of the technique, perhaps for later, rather than an immediate necessity. Finding the greatest multiple less than the dividend is maximally efficient, but being maximally efficient doesn't matter when you're just trying to get to grips with an idea and figure out what is true mathematically.

Writing as fractions

For older children who are becoming comfortable with addition and subtraction of fractions with the same denominator, writing division using fractions becomes a possibility. Is this harder, more formal or more abstract than the standard division algorithm? It feels to me that it might help to embed some important ideas about fractions, such as that $\frac{a}{d} + \frac{b}{d} = \frac{a + \mathrm{b}}{d}$ and not $\frac{a + \mathrm{b}}{2d}$, and that a fraction is just a number, and could be equal to an integer (e.g. $\frac{26}{13} = 2$). The notation works just as well, even if the divisor is something harder:

$$\frac{936}{13} = \frac{900}{13} + \frac{30}{13} + \frac{6}{13}$$
$$= \frac{910}{13} + \frac{20}{13} + \frac{6}{3}$$
$$= 70 + \frac{26}{13}$$
$$= 70 + 2$$
$$= 72$$

I would argue that this isn't an *alternative* to long division but *is* the process of long division, just written out differently. When people say 'long division', they generally mean both the process and a particular layout for it.

I prefer to think of the challenge of teaching short or long division as centred on one issue: What is a remainder? I don't think that the reason that long division may be difficult should be that there are so many steps to get right, with each step needing to be learned separately and then sequenced back together, so that the child gets each one right and in the right order. I suspect that if you instead spend the same amount of time working on 'What is a remainder?', then long division begins to make sense. And it then becomes obvious what you have to do. And then (finally, at the end) learning some conventional preferred layout for it is trivial.

Note

1. Of course, 48 itself is a multiple of 6, just as 45 was a multiple of 5. But the child didn't yet know their tables well, and weren't yet familiar with divisions that lead to a remainder. So they did not have much of an idea which multiple of 6 it might be.

Reference

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