Rationalising all Kinds of Denominators

By **Colin Foster**

Certain topics can seem rather 'procedural', and it can be difficult to find ways to extend the challenge for students who are successful with them, without just using harder numbers. For example, students may be able to rationalise the denominator in a fraction such as

$$\frac{3-\sqrt{2}}{\sqrt{2}+1}$$

by writing (Note 1):

$$\begin{aligned}\frac{3-\sqrt{2}}{\sqrt{2}+1} &= \left(\frac{3-\sqrt{2}}{\sqrt{2}+1}\right)\left(\frac{\sqrt{2}-1}{\sqrt{2}-1}\right)\\ &= \frac{\left(3-\sqrt{2}\right)\left(\sqrt{2}-1\right)}{2-1}\\ &= \frac{3\sqrt{2}-3-2+\sqrt{2}}{1}\\ &= 4\sqrt{2}-5.\end{aligned}$$

It's possible to generate lots of practice in topics like this by asking students to create their own examples. For instance, you could ask them to invent other fractions with irrational denominators that are also equal to $4\sqrt{2}-5$. The process of trial and error can lead to valuable practice, whether or not students' attempts turn out to equal the desired $4\sqrt{2}-5$. In the process, they begin to unpick the procedure to understand, for example, what must happen for the denominator to simplify to 1. When students succeed, you can offer the challenge of creating the most complicated-looking example that they can, or using certain numbers, such as 13, somewhere in their starting fraction, or only using the number 2 throughout. I call these sorts of tasks *mathematical études* (Foster, 2018).

How might this topic be further extended? What if we began instead with a fraction containing a *cube* root in the denominator, such as

$$\frac{1}{\sqrt[3]{2}+1} \quad \text{(Note 2).}$$

This looks superficially like a similar problem. But can we 'rationalise the denominator' in a case like this? What about for other, similar-looking fractions? If you haven't previously considered this, you may wish to pause to think about it before reading on.

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Doing this might sound impossible, because there is no 'difference of two cubes' to match the 'difference of two squares' that we rely on when rationalising a denominator containing a square root. However, it is certainly possible.

Let $r = \sqrt[3]{2} + 1$, the irrational denominator of our fraction. We require $\frac{1}{r}$ in a simplified form, with a rational denominator.

We rearrange as

$$\sqrt[3]{2} = r - 1,$$

and, cubing both sides; this gives

$$\begin{aligned}2 &= (r-1)^3\\ 2 &= r^3 - 3r^2 + 3r - 1.\end{aligned}$$

Students might think that we are stuck at this point, because they assume that they need to isolate r and make it the subject of the equation. But that isn't necessary. Expressing $\frac{1}{r}$ in terms of other rs is fine, because we can just substitute $r = \sqrt[3]{2} + 1$ for those other rs at the end. So, we have

$$\begin{aligned}3 &= r^3 - 3r^2 + 3r\\ \frac{3}{r} &= r^2 - 3r + 3\\ \frac{1}{r} &= \frac{r^2 - 3r + 3}{3}\end{aligned}$$

We have now rationalised the denominator, because we have a denominator of 3, and we can substitute $\sqrt[3]{2} + 1$ for each remaining r in the numerator, to obtain

$$\begin{aligned}\frac{1}{r} &= \frac{(\sqrt[3]{2}+1)^2 - 3(\sqrt[3]{2}+1) + 3}{3}\\ &= \frac{(\sqrt[3]{2})^2 + 2\sqrt[3]{2} + 1 - 3\sqrt[3]{2} - 3 + 3}{3}\\ &= \frac{(\sqrt[3]{2})^2 - \sqrt[3]{2} + 1}{3}.\end{aligned}$$

The numerator looks messy, but that doesn't matter: the denominator is rational, which is all that we wanted. It is reassuring to check on a calculator that

$$\frac{(\sqrt[3]{2})^2 - \sqrt[3]{2} + 1}{3} = \frac{1}{\sqrt[3]{2}+1},$$

because it isn't easy to spot, just by looking, whether these sorts of things 'look right' or not. Students can create problems like this for each other, and check their simplifications (at least approximately) on their calculators.

Another nice way to extend this topic is to explore *nested surds*, such as $\sqrt{2\sqrt{2}+3}$. If you evaluate this on a calculator, you get 2.414213562... Does this look a bit familiar? Could it possibly be $\sqrt{2}+1$ in disguise? It doesn't particularly *look* like $\sqrt{2}+1$, but appearances can be deceptive.

We assume (tentatively):

$$\sqrt{2\sqrt{2}+3}=\sqrt{p}+\sqrt{q}\,,$$

where p, $q > 0$. If supposing that $\sqrt{2\sqrt{2}+3}$ can be written as the sum of two roots isn't correct, then we should reach some contradiction at some point, such as finding that $p+1=p$, say, and that will tell us that $\sqrt{2\sqrt{2}+3}$ *doesn't* take this form. But let's see.

We begin by squaring both sides to obtain

$$2\sqrt{2}+3=p+2\sqrt{pq}+q,$$

For this to hold, $pq=2$ and $p+q=3$. These equations are symmetrical in p and q, because our original expression, $\sqrt{p}+\sqrt{q}$, was symmetrical. This is a good check that we haven't (hopefully) made an error. So, the two solutions ($p=1$, $q=2$ and $p=2$, $q=1$) are essentially the same solution. So, we can write

$$\sqrt{2\sqrt{2}+3}=\sqrt{2}+\sqrt{1}=\sqrt{2}+1$$

as suspected.

Now that we have seen that it is true, we can perhaps get there more elegantly:

$$\begin{aligned}\sqrt{2\sqrt{2}+3}&=\sqrt{1+2\sqrt{2}+2}\\&=\sqrt{(1+\sqrt{2})^2}\\&=1+\sqrt{2}.\end{aligned}$$

Another way to 'show that' something like $\sqrt{2\sqrt{2}+3}=1+\sqrt{2}$ is true is to square both sides and see that the same result $(1+2\sqrt{2}+2)$ is obtained (Note 3). Doing this, we can now begin to see that it is 'obvious' that these expressions are equal!

An alternative method that I often see used involves pre-empting the $\sqrt{2}$ portion of the answer from the beginning, by writing:

$$\sqrt{2\sqrt{2}+3}=a+b\sqrt{2}\,.$$

The argument would be that these expressions are clearly 'all about' multiples of $\sqrt{2}$. It would be ridiculous to suppose a $\sqrt{3}$ in the second term, for example.

Assuming that the expression can be put into this form, squaring gives

$$2\sqrt{2}+3=a^2+2ab\sqrt{2}+2b^2.$$

However, this now becomes a bit cumbersome, because we obtain the equations:

$$2ab=2,\text{ so } ab=1$$

and $a^2+2b^2=3$,

and these are messier to solve than the equations that we obtained with the form $\sqrt{p}+\sqrt{q}$. Because our unknowns are now at two different levels (*not*-multiplied-by-$\sqrt{2}$ and multiplied-by-$\sqrt{2}$), we have lost the symmetry. Now, we have to square the $ab=1$ equation, which introduces extraneous negative solutions, which we will then have to exclude later on. I think it is normally much easier with problems like this to write $\sqrt{p}\pm\sqrt{q}$ in every case. We are punished for being too specific and 'helpfully' including the $\sqrt{2}$.

It's worth working through to see what happens if you try something like

$$\sqrt{2\sqrt{2}+3}=a+b\sqrt{3}\,,$$

which includes the unlikely $\sqrt{3}$, rather than $\sqrt{2}$. We would be astonished if this worked! This time, squaring gives

$$2\sqrt{2}+3=a^2+2ab\sqrt{3}+3b^2,$$

and the two equations that we have to solve are now

$$a^2+3b^2=3.$$

and $$ab\sqrt{3}=\sqrt{2},\text{ so } ab=\sqrt{\frac{2}{3}}.$$

If a and b are rational, then ab can't be equal to $\sqrt{\frac{2}{3}}$, and therefore there are no values of a or b which will satisfy

$$\sqrt{2\sqrt{2}+3}=a+b\sqrt{3}.$$

We are never going to get a combination of integers and $\sqrt{2}$s to equal something to do with $\sqrt{3}$. This is for the same reason that any claimed equation linking $\sqrt{2}$ and $\sqrt{3}$ only via addition and multiplication of integers, such as

$$5\sqrt{2}+4\sqrt{3}=14,$$

can only ever be approximately true.

There is much for students to explore here, which gives opportunities to review solving simultaneous equations, quadratic equations and honing their algebraic skills. They also have the opportunity to behave like mathematicians in asking and answering their own questions and developing a reasoned argument to justify their conclusions.

Notes

1. In this case, the denominator is not just *rationalised* (made rational) but it has become an integer ('integerised'?), which happens to also be unity ('unitised'?). An extension task can be for students to devise examples in which the denominator disappears (i.e. becomes equal to 1) like this.
2. I am grateful to Posamentier & Salkind (1996, pp. 236-237, §20-17) for a related problem that got me thinking about this.
3. Of course, the squares of two expressions being equal doesn't necessarily mean that the original two expressions were equal: one could have been the negative of the other. But, in this case, we know that both expressions are strictly positive, so this caution doesn't apply.

References

Foster, C. 2018 'Developing mathematical fluency: Comparing exercises and rich tasks', *Educational Studies in Mathematics*, 97(2), pp. 121–141. **https://doi.org/10.1007/s10649-017-9788-x**

Posamentier, A. S., & Salkind, C. T. 1996 *Challenging Problems in Algebra*. Courier Corporation.

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