Closed but provocative questions: curves enclosing unit area

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This article describes a task leading to work on curve sketching, simultaneous equations and integration to find the area enclosed between two curves. An initial closed question is used to confront students with a provocative answer, which they then explore in a much more open-ended way.

Keywords: closed questions; integration; inverse problems; open questions; provocative questions; working backwards

Sometimes mathematics teachers are told that open questions (those with more than one correct answer) are better than closed questions (those with one right answer), but this is too simplistic. Some closed questions can have answers that are so provocative that they invite further exploration or demand an explanation, leading to rich mathematical thinking.[1] I often like to offer students a closed starting point, which draws on routine procedures that they have learned, but which leads to a provocative result which motivates them to go on to pursue related mathematics in a more exploratory way.[2,3]

Here is an example.

Find the area enclosed between the curves $y = 4x^2 - 18x + 22$ and $y = -2x^2 + 12x - 14$.

On the face of it, this seems like a perfectly standard textbook question – but the outcome is striking. These curves intersect when $4x^2 - 18x + 22 = -2x^2 + 12x - 14$; i.e. when $6x^2 - 30x + 36 = 0$. So $x^2 - 5x + 6 = 0$, so $(x - 2)(x - 3) = 0$, so at (2, 2) and (3, 4).

So

$$\text{Area} = \int_2^3 \left(-6x^2 + 30x - 36\right) dx = \left[-2x^3 + 15x^2 - 36x\right]_2^3 = 1.$$

The area between these curves is 1 (Figure 1).

Although we have now answered the initial closed question, clearly we have only just begun. How did we set this up to come out so nicely? (Sometimes students will express surprise or ask, ‘How did you make that happen?’) The main task then becomes to create more examples where two curves enclose unit area and to see what we can find out about how to go about it. Encouraging students to generate their own examples to satisfy a mathematical constraint is a powerful pedagogical approach.[4]

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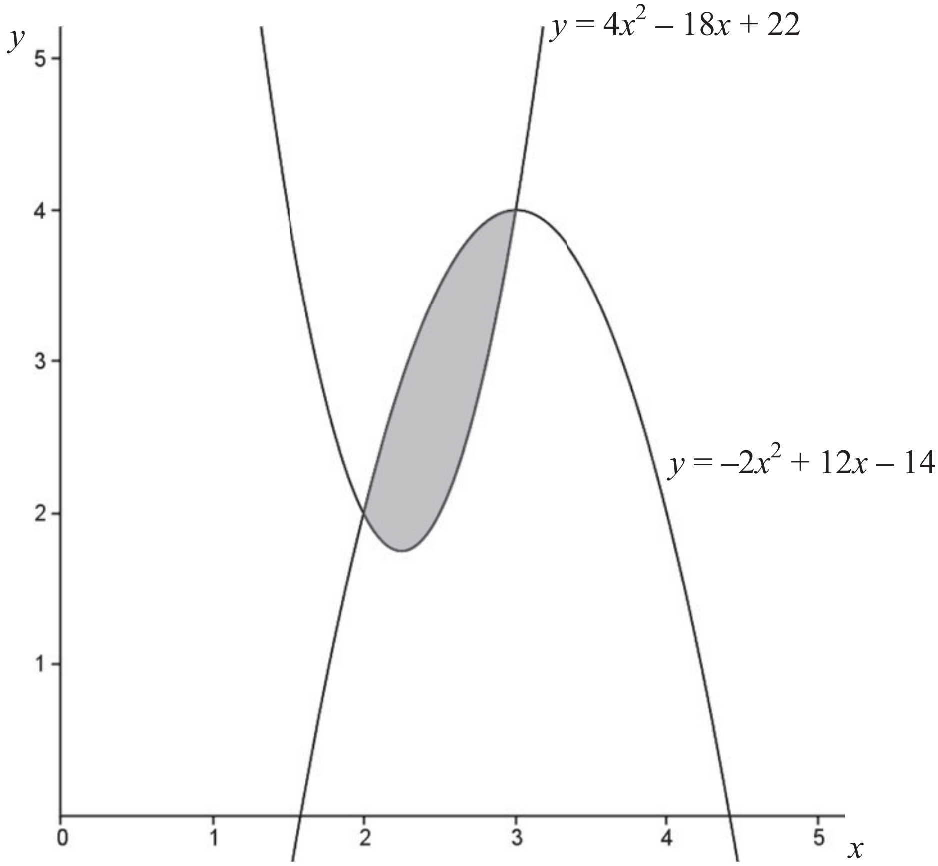

Figure 1. Two curves enclosing unit area.

One possible way to generate more examples would be to translate these two curves by the same vector. For example, if we let $x \to x + a$ and $y \to y + b$ then we will obtain as many pairs of curves as we like enclosing a region congruent to that shown in Figure 1, and therefore of unit area. Likewise we could scale horizontally and vertically by factors k and $1/k$, where $k > 0$, and obtain stretched versions of these curves, still enclosing unit area. This may be useful thinking for students but is perhaps not particularly exciting. With this sort of task, it is not enough just to find examples – students need to be challenged to find 'surprising' or 'impressive' examples!

A simple way to start might be to investigate the area enclosed between a single parabola and the x-axis, so it seems worth exploring integrals of the form $\int_a^b -(x-a)(x-b)\,dx$, where $a < b$. Geometrically, this integral will be equal to the area enclosed between the curve $y = (x - a)(x - b)$ and the x-axis. So we can calculate

$$\int_a^b -(x-a)(x-b)\,dx = \int_a^b \left(-x^2 + (a+b)x - ab\right)dx$$

$$= \left[-\frac{x^3}{3} + \frac{(a+b)x^2}{2} - abx \right]_a^b = \frac{1}{6}(-2(b^3 - a^3)$$
$$+3(a+b)\left(b^2 - a^2\right) - 6ab(b-a))$$
$$= \frac{1}{6}(b-a)\left(-2\left(b^2 + ab + a^2\right) + 3(a+b)^2 - 6ab\right) = \frac{1}{6}(b-a)^3.$$

This gives us the useful result that

$$\frac{6}{(a-b)^3} \int_a^b (x-a)(x-b)\,dx = 1, \text{ when } a < b.$$

If we want to avoid fractional coefficients in the equations of our curves, we can set $b - a = 1$, so, for example, $a = 1$ and $b = 2$, to give

$$\int_1^2 -6(x-1)(x-2)dx = 1 \quad (^*)$$

So the integrand here is $-6x^2 + 18x - 12$. This means that the area enclosed between the curve $y = -6x^2 + 18x - 12$ and the x-axis is 1 (Figure 2).

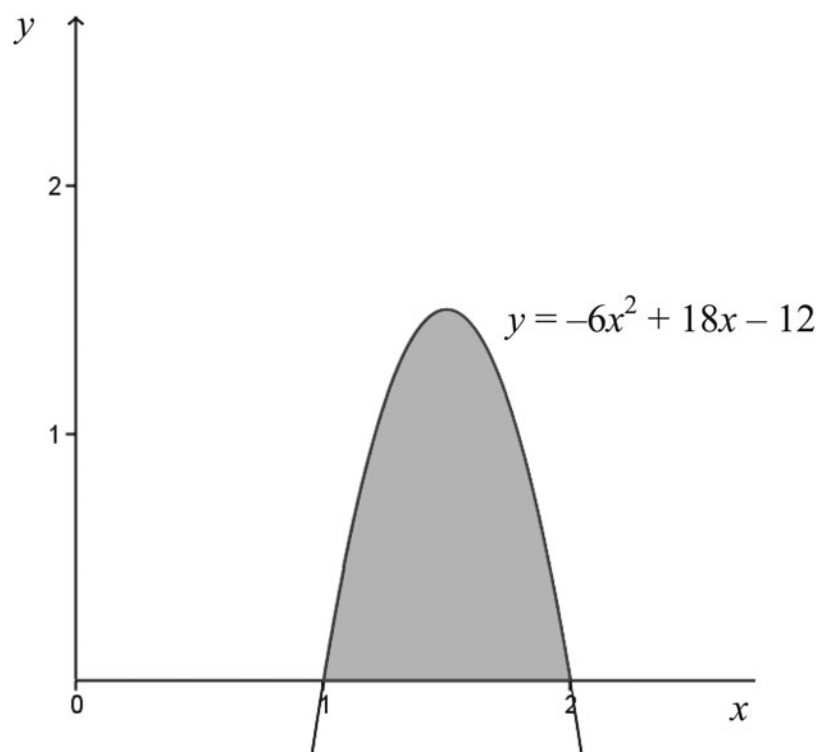

Figure 2. A parabola enclosing unit area above the x-axis.

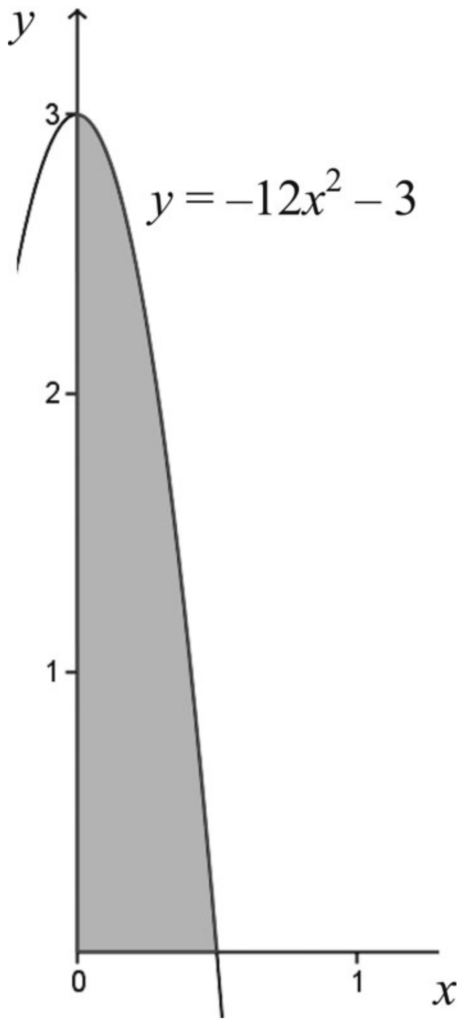

Figure 3. A parabola enclosing unit area in the first quadrant.

As before, we can consider simple transformations of this solution. Clearly, it would be possible to translate this curve horizontally by letting $x \to x + a$ and thus obtain as many congruent regions as we like between the curve and the x-axis, all enclosing unit area. A slightly different possibility would be to translate the curve $1\frac{1}{2}$ units to the left and then stretch it by a factor of 2 in the y-direction, to obtain $y = -12x^2 - 3$. The 'half' quadratic bounded above by this curve, below by the x-axis and to the left by the y-axis will enclose unit area (Figure 3).

Less symmetrical solutions within the first quadrant (making use of both axes as boundaries), such as $y = \frac{1}{9}(x + 1)(3 - x)$ (Figure 4), are also possible. These can be obtained by choosing any arbitrary quadratic with one positive root α and one negative root, integrating from $x = 0$ to α to obtain I, and then scaling the quadratic by $1/I$, thus giving a region with unit area. It is also possible to obtain such solutions by simplifying $-\int_0^b k(x - a)(x - b)dx = 1$, where $a < 0$ and $b > 0$, to obtain $k = \frac{6}{b^2(b-3a)}$. (Note that

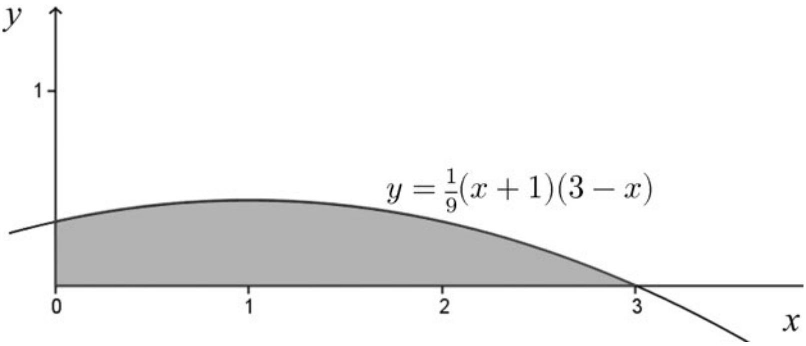

Figure 4. A parabola enclosing unit area in the first quadrant.

$b \neq 3a$, since a and b are of opposite sign.) Choosing the values $a = -1$ and $b = 3$, for instance, gives $k = \frac{1}{9}$ and the solution $y = \frac{1}{9}(x + 1)(3 - x)$ stated above and shown in Figure 4.

However, there is no reason to restrict regions to being bounded by the axes. Since we know from (*) that $\int_1^2 -6(x-1)(x-2)\,dx = 1$, any pair of functions with a difference of $-6x^2 + 18x - 12$, such as the parabola $y = 6x^2$ and the line $y = 18x - 12$, will enclose

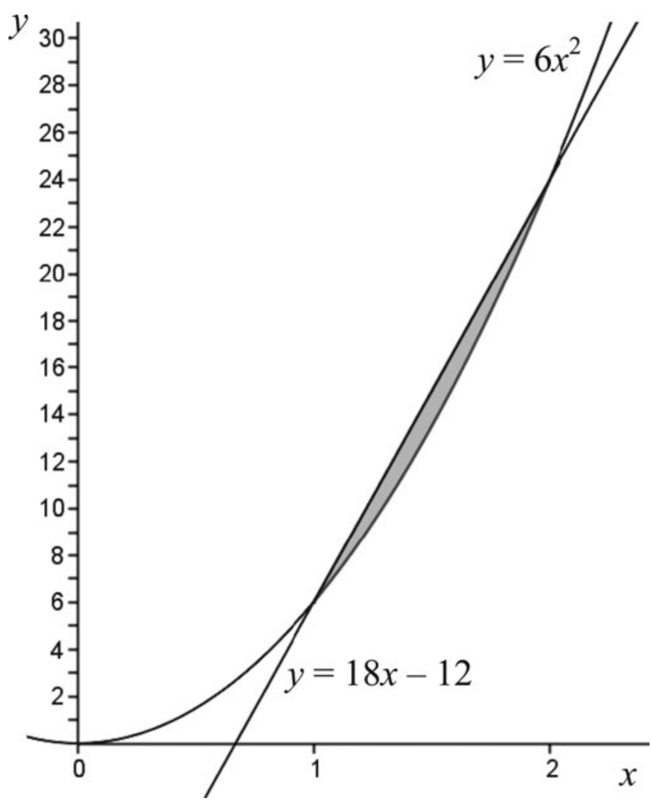

Figure 5. A line and a parabola enclosing unit area.

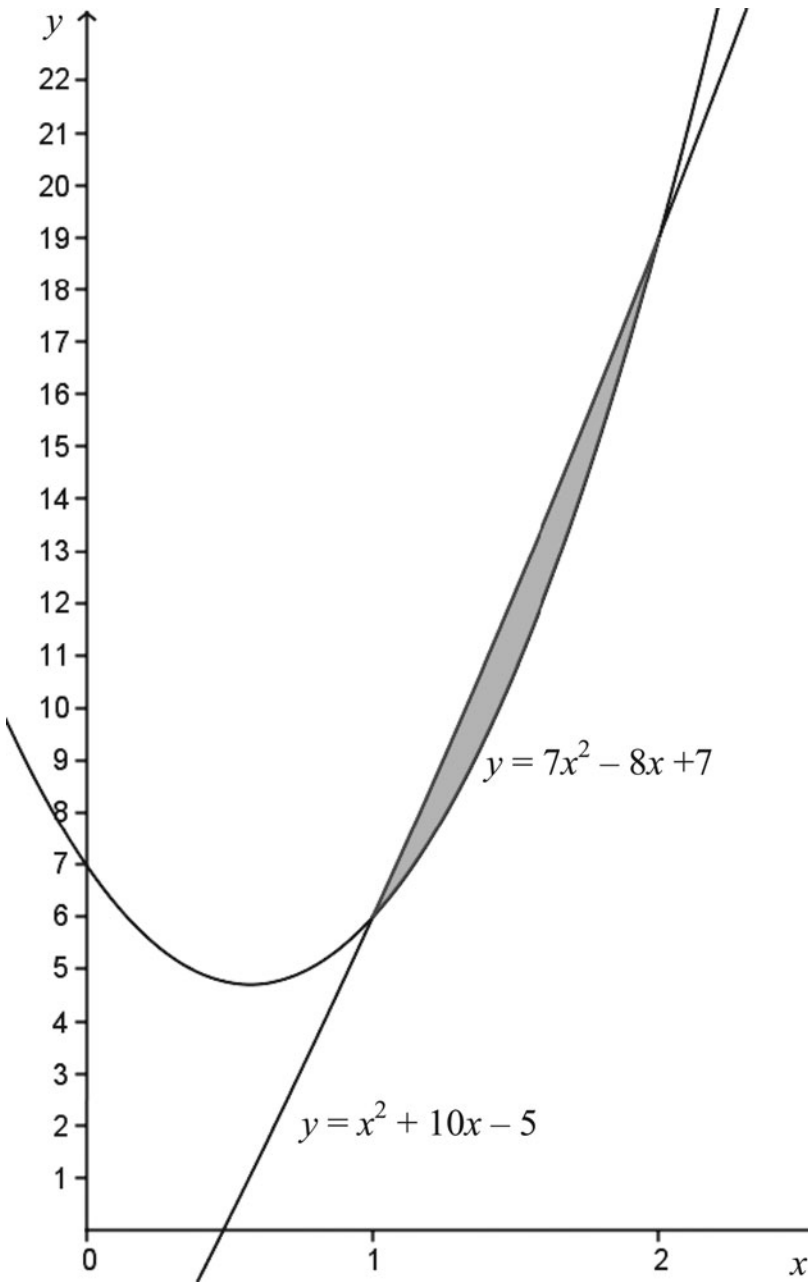

Figure 6. Two parabolas (the same way up) enclosing unit area.

unit area, as shown in Figure 5. In a similar way, we can use our result (*) to find *pairs* of quadratics that will work. We simply choose two quadratic expressions with a difference of $-6x^2 + 18x - 12$, such as $y = x^2 + 10x - 5$ and $y = 7x^2 - 8x + 7$, and these graphs will enclose unit area, as shown in Figure 6. (Students may be surprised that quadratic curves 'the same way up' – i.e. with coefficients of x^2 having the same sign, can enclose a finite area.)

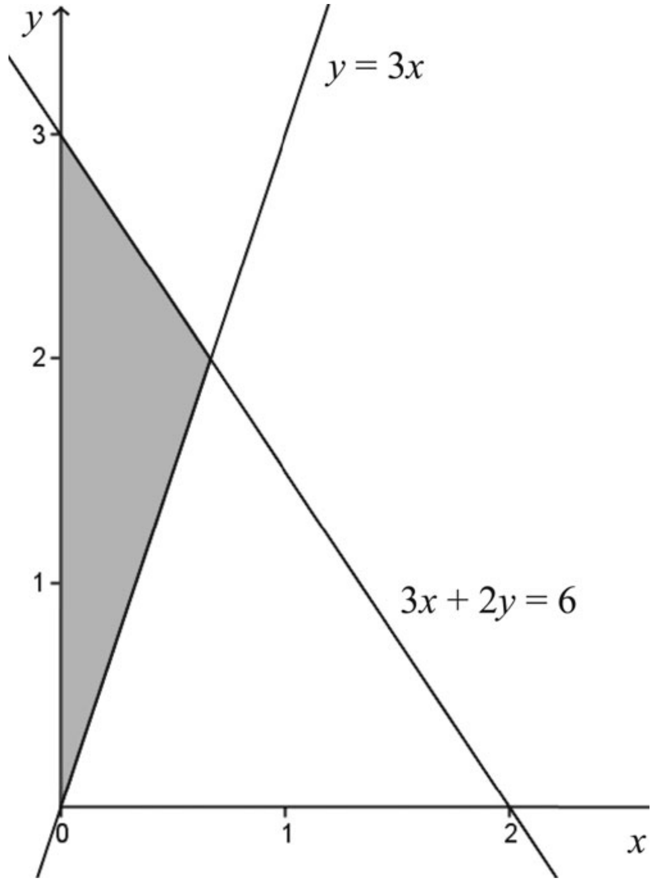

Figure 7. Two lines and the y-axis enclosing unit area.

Clearly there are many other directions in which students might go from the given starting point. For example, they might consider regions bounded between two lines and an axis, such as in Figure 7, where the area of the shaded triangle may be seen as the difference between a right-angled triangle of area 3 and a scalene triangle of area 2. While not requiring integration in this case, some cases may call on careful reasoning involving similar triangles, or the solution of simultaneous equations. For instance, students might consider the question: Can three lines specified by equations with integer coefficients, none of which is horizontal or vertical, enclose a triangle of unit area? Alternatively they might explore cubics or hyperbolae. In this way, the task is open enough to support productive exploration at a variety of levels.

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References

[1] Foster C. Fitting shapes inside shapes: closed but provocative questions. Math Sch. Forthcoming 2015.

[2] Foster C. The convergentdivergent model: an opportunity for teacher–learner development through principled task design. Educ Designer. Forthcoming 2015.
[3] Foster C. Mathematical études: embedding opportunities for developing procedural fluency within rich mathematical contexts. Int J Math Educ Sci Technol. 2013;44(5):765–774.
[4] Watson A, Mason J. Mathematics as a constructive activity: learners generating examples. Mahwah, NJ: Erlbaum; 2005.

Effectively using multiple technologies

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Details regarding three examples where students were engaged in mathematics reasoning in effective and productive ways are shared. Technology in these instances allowed for students to share their thinking in ways that might not have otherwise been possible. These examples also serve to show how the choice of technology influences the mathematics to be learned from a mathematical task.

Keywords: technology; mathematics; mathematics education

1. Introduction

The use and choice of technology in the classroom as a conduit for mathematics learning must be purposeful. Planning for technological opportunities, that engage all students, while also accomplishing the learning goals of the enacted lesson are important. To that end, I have found that the use of mathematics technology, along with effective and engaging mathematical problems that are of a high cognitive demand,[1] allow students to communicate and share mathematical thinking in ways that may not have been previously possible for them. Technology also allows me to better understand what my students understand.

Additionally, the National Council of Teachers of Mathematics have also indicated their strong support for technology use in teaching and learning mathematics through their Tools and Technology Principle,[2] which states:

> 'An excellent mathematics program integrates the use of mathematical tools and technology as essential resources to help students learn and make sense of mathematical ideas, reason mathematically, and communicate their mathematical thinking'. [2,p.5]

To illustrate how these ideas have played out in practice, I share several examples where the structure of the task and the use of technology allowed students to engage in effective and productive mathematical thinking (examples come from courses I have taught for future secondary teachers). More importantly, the use of different technologies highlights the different mathematical ideas and levels of understanding that can be accessed due to the choices of technology.

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